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Comparison of Generalized Approximate Cruise Range Solutions for Turbojet/Fan Aircraft

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Two different approximate solutions are proposed and compared with the numerical solution of the constant-altitude–constant-Mach-number cruise range for turbojet/fan aircraft. The approximate solutions consider both cambered wing drag polar and dependence of specific fuel consumption on Mach number. It is also proved that the classical analytical approach assuming constant specific fuel consumption and uncambered wing drag polar results in higher optimum Mach numbers and shorter ranges for constant-altitude–constant-airspeed flight of turbojet/fan transport aircraft.

Nomenclature

a	=	speed of sound
C_D	=	drag coefficient
C_{D_0}	=	parasite drag coefficient
C_L	=	lift coefficient
c_T	=	thrust specific fuel consumption
c_{T_0}	=	reference thrust specific fuel consumption for flight altitude
D	=	drag force
E_m	=	maximum lift-to-drag ratio with cambered wing
E_{m0}	=	maximum lift-to-drag ratio without cambered wing
K	=	range parameter
k_1, k_2	=	induced drag coefficients
L	=	lift force
M	=	mach number
M_{md_0}	=	minimum drag Mach number
$\bar{M}$	=	average Mach number
m	=	ratio of the Mach number to the minimum drag Mach number
m_{sa}	=	ratio of the Mach number to the minimum drag Mach number for small angle approximation
p	=	pressure at flight altitude
$\Re$	=	nondimensional range
$\Re_{\mathrm{sa}}$	=	nondimensional range for small angle approximation
$\bar{\Re}$	=	nondimensional range for average Mach number approximation
S	=	wing area
W	=	weight of aircraft
W_0	=	aircraft weight at the beginning of cruise flight
W_1	=	aircraft weight at the end of cruise flight
w	=	ratio of the aircraft weight to the initial condition of weight
X	=	range
β	=	thrust specific fuel consumption Mach number exponent
γ	=	specific heat ratio for air
ζ	=	fuel weight fraction

Introduction

RANGE is a basic operational requirement and design criterion especially for transport category aircraft. Because of this fact, range performance of aircraft has been studied by several authors since the first days of powered flight. Recently, Torenbeek[1] reviewed cruise performance and range prediction in detail. According to Torenbeek, the first analyses of range for turbojet-powered aircraft were published soon after Second World War, by Page and Edwards in the United Kingdom and by Jonas, Askenas, and Perkins and Hage in the United States. The range equations introduced with these studies were based either on a cruise/climb flight with constant speed and lift coefficient, resulting in the Bréguet equation for jet propulsion, or on constant-altitude flight with constant lift coefficient. Later on, some other cruise flight procedures like constant-altitude–constant-airspeed or-Mach-number, constant-altitude–constant-thrust, constant-lift-coefficient–constant-thrust, and constant-airspeed–constant-thrust flight were developed. Anathasayanam[2] summarizes all of these techniques.

Although the cruise/climb technique results in longer ranges than horizontal cruising flight, it is not usually a favored operational procedure due to air-traffic-control-imposed constraints.[1] Generally, when flight is conducted under the jurisdiction of air traffic control regulations, the accepted cruise flight is the constant-altitude–constant-airspeed or -Mach-number procedure.[3] In this Paper, solutions of cruise range equation for turbojet/fan aircraft are studied for the constant-altitude–constant-Mach-number condition. Because this condition leads to a range equation that is an arctangent function, optimum Mach number and maximum range are usually solved analytically by constant specific fuel consumption and uncambered wing drag polar assumptions. Previously, Bert[4] proposed range and endurance solutions for aircraft having wings with camber but with constant specific fuel consumption. On the other hand, Martinez-Val[5] proposed introduction of the dependence of specific fuel consumption on Mach number but with uncambered wing polar assumption. In this Paper, two approximate solutions, considering both thrust specific fuel consumption variation with Mach number and aircraft with cambered wing, are proposed and compared with the numerical solution of the arctangent range function.

Problem Formulation

The constant-altitude–constant-airspeed (Mach number) range of a jet aircraft can be determined from

$$dX = -(aM/c_T)(L/D)(dW/W) \tag{1}$$

where c_T is the thrust specific fuel consumption. At constant altitude, the thrust specific fuel consumption varies with Mach number as[5]

$$c_T = c_{T_0} M^{\beta} \tag{2}$$

On the other hand, for cambered wing drag polar,

$$C_D = C_{D_0} + k_1 C_L^2 - k_2 C_L \tag{3}$$

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Although all coefficients of the drag polar are functions of Mach number, they will be assumed constant. The errors involved in Eq. (3) due to this assumption are very small in long-range cruise, because this condition is not affected by drag divergence.[5] Lift-to-drag ratio in Eq. (1) can be written in terms of weight and Mach number:

$$\frac{L}{D} = 2E_{m_0}\frac{m^2 w}{m^4 - 2k_2 E_{m_0} w m^2 + w^2} \tag{4}$$

where E_m is the maximum lift-to-drag ratio,

$$E_m = 1\Big/\left(2\sqrt{k_1 C_{D_0}} - k_2\right), \qquad E_{m_0} = 1\Big/2\sqrt{k_1 C_{D_0}} \tag{5}$$

w is the ratio of aircraft weight to W_0, aircraft weight at the beginning of cruise flight,

$$w = W/W_0$$

and

$$m = M/M_{\mathrm{md}_0} \tag{6}$$

where M_{md_0} is the minimum drag Mach number for initial conditions of cruise flight:

$$M_{\mathrm{md}_0} = \left(k_1\big/C_{D_0}\right)^{\frac{1}{4}}(2W_0/\gamma p S)^{\frac{1}{2}} \tag{7}$$

To ensure that flight is not in a statically unstable equilibrium condition, use $m \geq 1$. Integrating Eq. (1) between the aircraft weight at the beginning, W_0, and the aircraft weight at the end, W_1, of the cruise flight yields nondimensional range:

$$\Re = \frac{X c_{T_0}}{2aM_{\mathrm{md}_0}^{1-\beta}}\sqrt{\frac{1+k_2E_{m_0}}{E_m E_{m_0}}} = m^{1-\beta}\left[\tan^{-1}\sqrt{\frac{E_m}{E_{m_0}}}\frac{1-k_2E_{m_0}m^2}{m^2\sqrt{1+k_2E_{m_0}}} - \tan^{-1}\sqrt{\frac{E_m}{E_{m_0}}}\frac{(1-\zeta)-k_2E_{m_0}m^2}{m^2\sqrt{1+k_2E_{m_0}}}\right] \tag{8}$$

or

$$\Re = m^{1-\beta}\left[\tan^{-1}\sqrt{\frac{E_{m_0}}{E_m}}\frac{\zeta m^2\sqrt{1+k_2E_{m_0}}}{m^4-(2-\zeta)k_2E_{m_0}m^2+(1-\zeta)}\right] \tag{9}$$

where ζ is the cruise fuel weight fraction:

$$\zeta = (W_0 - W_1)/W_0 \tag{10}$$

For constant altitude, maximizing the range implies that

$$\frac{\partial\Re}{\partial m} = 0 \tag{11}$$

The result obtained from Eq. (11) is

$$(1-\beta)\left[\tan^{-1}\sqrt{\frac{E_{m_0}}{E_m}}\frac{\zeta m^2\sqrt{1+k_2E_{m_0}}}{m^4-(2-\zeta)k_2E_{m_0}m^2+(1-\zeta)}\right] - \left\{\frac{2\zeta\sqrt{E_{m_0}\big/E_m}\sqrt{1+k_2E_{m_0}}[m^4-(1-\zeta)]m^2}{\left[m^4-(2-\zeta)k_2E_{m_0}m^2+(1-\zeta)\right]^2+\zeta^2E_{m_0}\big/E_m\left(1+k_2E_{m_0}\right)m^4}\right\} = 0 \tag{12}$$

This equation does not yield an analytic expression for m because of the arctangent function. However, some approximate solutions can be proposed in order to determine the Mach number yielding maximum range for constant altitude. Two approximate solutions are given hereafter.

Approximate Solution by Small Angle Assumption

For optimum cruising the angle defined by the arctangent term is usually small and arctangent can be set equal to its argument as was proposed by both Torenbeek[1] and Ojha.[6] This approximation results in the following:

$$\Re = \sqrt{\frac{E_{m_0}}{E_m}}\frac{\zeta m^{3-\beta}\sqrt{1+k_2E_{m_0}}}{m^4-(2-\zeta)k_2E_{m_0}m^2+(1-\zeta)} \tag{13}$$

Application of the maximum cruise range condition given by Eq. (11) to this accurate approximation[1] results in the following:

$$m_{\mathrm{sa}}^2 = \frac{-(1-\beta)k_2E_{m_0} + \sqrt{(1-\beta)^2k_2^2E_{m_0}^2 + 4[(1+\beta)(3-\beta)(1-\zeta)/(2-\zeta)^2]}}{1+\beta}\,\frac{2-\zeta}{2} \tag{14}$$

By substituting this optimum Mach number in Eq. (9), an approximate maximum constant-altitude–constant-speed range can be found:

$$\Re_{\mathrm{sa}} = m_{\mathrm{sa}}^{1-\beta}\left[\tan^{-1}\sqrt{\frac{E_{m_0}}{E_m}}\frac{\zeta m_{\mathrm{sa}}^2\sqrt{1+k_2E_{m_0}}}{m_{\mathrm{sa}}^4-(2-\zeta)k_2E_{m_0}m_{\mathrm{sa}}^2+(1-\zeta)}\right] \tag{15}$$

A simplified solution for optimum Mach number and maximum range can be found by assuming that aircraft wing has no camber, $k_2 = 0$, and thrust specific fuel consumption is constant; then $\beta = 0$. For this special case,

$$m_{\mathrm{sa}} = [3(1-\zeta)]^{\frac{1}{4}} \tag{16}$$

This yields simplified maximum range as follows:

$$\Re_{\mathrm{sa}} = [3(1-\zeta)]^{\frac{1}{4}}\tan^{-1}\left(\sqrt{3}\zeta\big/4\sqrt{1-\zeta}\right) \tag{17}$$

Approximate Solution by Average Mach Number

For a given amount of fuel, maximizing the range is equal, as a first approximation, to maximizing the range parameter for each airplane weight during cruise[2]:

$$K = (M/c_T)(L/D) \tag{18}$$

After substituting Eqs. (2) and (4) into Eq. (18),

$$K = \left(w\big/c_{T_0}\right)\left[m^{3-\beta}\big/\left(m^4 - 2k_2E_{m_0}wm^2 + w^2\right)\right] \tag{19}$$

and maximizing the range parameter

$$dK/dm = 0 \tag{20}$$

results in

$$m^2 = \frac{-(1-\beta)k_2E_{m_0} + \sqrt{(1-\beta)^2k_2^2E_{m_0}^2 + (1+\beta)(3-\beta)}}{1+\beta}w \tag{21}$$

This Mach number represents only an instantaneous optimum within cruise-flight range. Because aircraft weight changes due to fuel consumption, this result proposes Mach-number change during cruise flight. However, to meet constant-altitude–constant-Mach-number flight condition, an average Mach number can be assumed between initial and final conditions of flight as follows:

$$\bar{m}^2 = \frac{m_0^2 + m_1^2}{2} = \frac{2-\zeta}{2(1+\beta)}\Bigg[-(1-\beta)k_2 E_{m_0} + \sqrt{(1-\beta)^2 k_2^2 E_{m_0}^2 + (1+\beta)(3-\beta)}\Bigg] \tag{22}$$

Substituting the Mach number found from Eq. (22) into Eq. (9) would result in an approximate maximum cruise range for constant-altitude–constant-airspeed flight conditions:

$$\bar{\Re} = \bar{m}^{1-\beta}\left[\tan^{-1}\sqrt{\frac{E_{m_0}}{E_m}}\frac{\zeta\bar{m}^2\sqrt{1+k_2 E_{m_0}}}{\bar{m}^4 - (2-\zeta)k_2 E_{m_0}\bar{m}^2 + (1-\zeta)}\right] \tag{23}$$

For the special case of a simplified solution, assuming that the aircraft wing has no camber, $k_2 = 0$, and thrust specific fuel consumption is constant, $\beta = 0$,

$$\bar{m}^2 = \left(\sqrt{3}/2\right)(2-\zeta) \tag{24}$$

Substituting Eq. (24) into Eq. (9) results in a simplified approximate maximum cruise range:

$$\bar{\Re} = 3^{\frac{1}{4}}\sqrt{\frac{2-\zeta}{2}}\tan^{-1}\left[2\sqrt{3}\frac{\zeta(2-\zeta)}{(4-\zeta)(4-3\zeta)}\right] \tag{25}$$

Comparison of Solutions

To demonstrate the accuracy of approximate solutions, a hypothetical aircraft is considered. The drag polar of the hypothetical aircraft is assumed as follows:

$$C_D = 0.025 - 0.0215 C_L + 0.083 C_L^2$$

Anathasayanam[2] states that the value of β varies between 0.4 and 0.7 for high-bypass-ratio turbofan engines. For hypothetical aircraft it is assumed that $\beta = 0.5$.

Table 1 Optimum Mach numbers given in m for various solutions

	Numerical solutions of Eq. (9)			Approximate solutions, $k_2 = 0.0215$, $\beta = 0.5$				Simplified solutions, $k_2 = 0$, $\beta = 0$			
	$\beta = 0.5$										
ζ	$k_2 = 0.0215$	$k_2 = 0$	Error, %	Small Angle	Error, %	Average Mach	Error, %	Small Angle	Error, %	Average Mach	Error, %
0.050	1.088	1.122	3.10	1.088	0.00	1.088	0.02	1.299	19.41	1.300	19.43
0.100	1.074	1.107	3.09	1.073	−0.02	1.074	0.05	1.282	19.39	1.283	19.47
0.150	1.059	1.092	3.08	1.058	−0.07	1.060	0.10	1.264	19.34	1.266	19.54
0.200	1.048	1.076	3.08	1.042	−0.13	1.046	0.20	1.245	19.28	1.249	19.65
0.250	1.028	1.059	3.08	1.025	−0.21	1.031	0.34	1.225	19.20	1.231	19.81
0.300	1.011	1.042	3.06	1.008	−0.33	1.016	0.51	1.204	19.07	1.213	20.02
0.321	1.004	1.035	3.07	1.000	−0.39	1.010	0.60	1.195	19.01	1.206	20.13
0.332	1.000	1.031	3.07	1.000	0.00	1.007	0.65	1.190	18.98	1.202	20.19
0.350	1.000	1.024	2.39	1.000	0.00	1.001	0.11	1.182	18.17	1.195	19.54
0.354	1.000	1.023	2.28	1.000	0.00	1.000	0.00	1.180	18.01	1.194	19.41
0.400	1.000	1.005	0.51	1.000	0.00	1.000	0.00	1.158	15.83	1.177	17.71
0.414	1.000	1.000	0.00	1.000	0.00	1.000	0.00	1.151	15.14	1.172	17.19
0.450	1.000	1.000	0.00	1.000	0.00	1.000	0.00	1.133	13.34	1.59	15.86
0.500	1.000	1.000	0.00	1.000	0.00	1.000	0.00	1.107	10.67	1.140	13.98

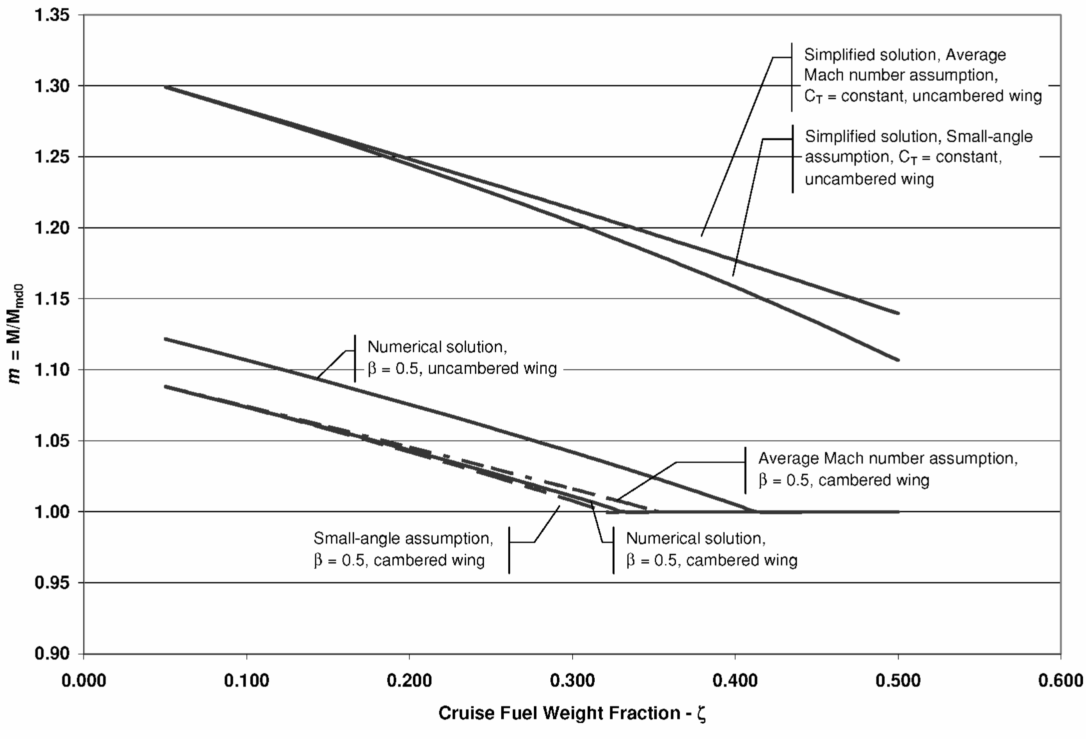

Fig. 1 Variation of $m = M/M_{\mathrm{md}_0}$ with cruise fuel weight fraction.

Table 2 Nondimensional range for various solutions

	Numerical solutions of Eq. (9)			Approximate solutions, $k_2 = 0.0215$, $\beta = 0.5$				Simplified solutions, $k_2 = 0$, $\beta = 0$			
	$\beta = 0.5$										
ζ	$k_2 = 0.0215$	$k_2 = 0$	Error, %	Small Angle	Error, %	Average Mach	Error, %	Small Angle	Error, %	Average Mach	Error, %
0.050	0.033	0.026	−20.78	0.033	0.00	0.033	0.00	0.029	−13.08	0.029	−13.08
0.100	0.068	0.054	−20.78	0.068	0.00	0.068	0.00	0.058	−13.64	0.058	−13.64
0.150	0.104	0.082	−20.76	0.104	0.00	0.104	0.00	0.089	−14.22	0.089	−14.22
0.200	0.141	0.112	−20.74	0.141	0.00	0.141	0.00	0.120	−14.81	0.120	−14.81
0.250	0.180	0.143	−20.71	0.180	0.00	0.180	0.00	0.152	−15.42	0.152	−15.42
0.300	0.221	0.175	−20.66	0.221	0.00	0.221	−0.01	0.185	−16.04	0.185	−16.04
0.321	0.238	0.189	−20.64	0.238	0.00	0.238	−0.01	0.199	−16.31	0.199	−16.30
0.332	0.248	0.197	−20.63	0.248	0.00	0.248	−0.01	0.207	−16.45	0.207	−16.45
0.350	0.264	0.209	−20.60	0.264	0.00	0.264	0.00	0.220	−16.68	0.220	−16.67
0.354	0.267	0.212	−20.59	0.267	0.00	0.267	0.00	0.222	−16.72	0.222	−16.72
0.400	0.308	0.245	−20.42	0.308	0.00	0.308	0.00	0.255	−17.23	0.255	−17.22
0.414	0.321	0.255	−20.35	0.321	0.00	0.321	0.00	0.265	−17.37	0.265	−17.36
0.450	0.354	0.283	−20.12	0.354	0.00	0.354	0.00	0.291	−17.68	0.291	−17.66
0.500	0.401	0.322	−19.76	0.401	0.00	0.401	0.00	0.329	−18.00	0.329	−17.96

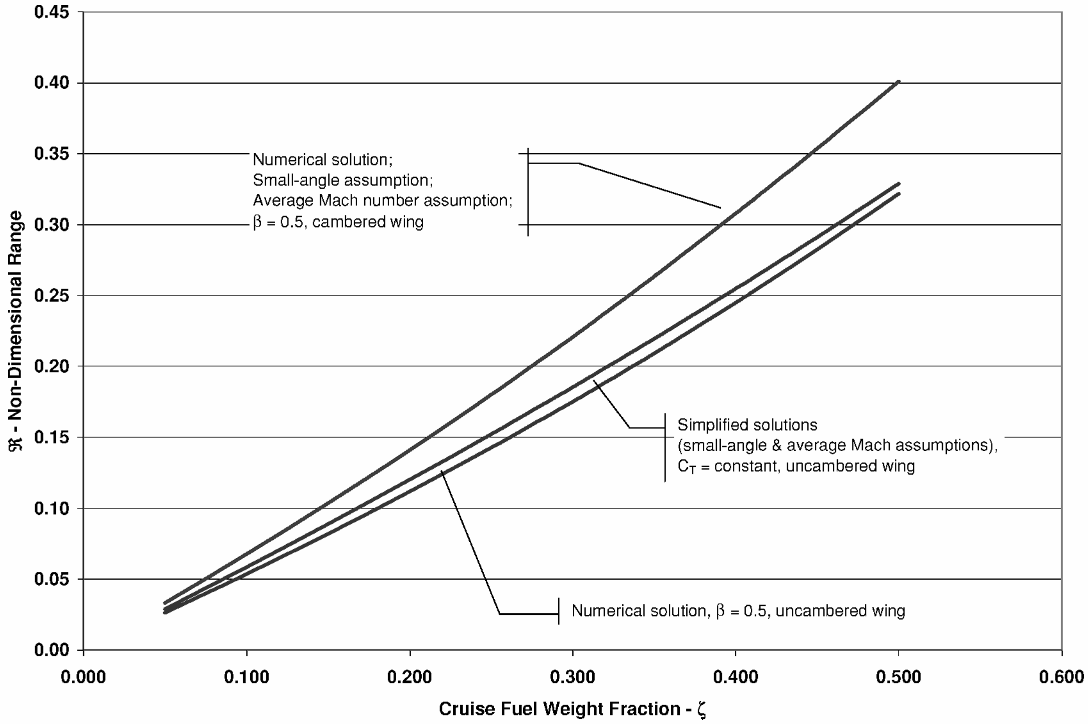

Fig. 2 Variation of nondimensional range with cruise fuel weight fraction.

The optimum Mach numbers for various fuel weight fractions found by small angle assumption [Eq. (14)] and average Mach number assumption [Eq. (22)] are given in Table 1 and Fig. 1. The optimum Mach numbers found by numerical solution of Eq. (9) both for cambered and uncambered wings are also given in Table 1 and Fig. 1 along with simplified solutions for small angle assumption [Eq. (16)] and average Mach number assumption [Eq. (24)]. The errors given in Table 1 show the percentage difference between approximate solutions and the numerical solution of Eq. (9) for $k_2 = 0.0215$ and $\beta = 0.5$.

The numerical treatment of Eq. (9) for the uncambered wing assumption results in about 3% higher optimum Mach numbers, although specific fuel consumption variation with Mach number is not neglected. The optimum Mach numbers found by small angle assumption [Eq. (14)] are slightly less than the ones found by numerical solution. Maximum difference for this case is only about −0.39%. The optimum Mach numbers found by average Mach number assumption [Eq. (22)] are slightly higher than numerical solution values. However, maximum difference is still less than 1%. Both simplified solutions assuming constant specific fuel consumption and uncambered wing result in about 20% higher optimum Mach numbers. The results also show that an aircraft with cambered wing has to fly at minimum drag Mach number for longer ranges corresponding to cruise fuel weight fraction above 0.33.

The nondimensional range values found by substitution of the optimum Mach numbers of small angle and average Mach number assumptions to Eq. (9) are given in Table 2 and Fig. 2. Besides these, maximum nondimensional range found by numerical treatment of Eq. (9) is also given together with the results of the numerical solution of Eq. (9) for the uncambered wing assumption. Both Table 2 and Fig. 2 also show the nondimensional range values calculated by substitution of the optimum Mach numbers of simplified solutions to Eq. (9).

The numerical solution of Eq. (9) for uncambered wing results in about 21% shorter nondimensional range. Simplified solutions of small angle and average Mach number assumptions yield about 16% shorter nondimensional range. Both small angle and average Mach number assumptions yield almost 0% difference from the numerical solution of Eq. (9).

Conclusions

It is proved that the classical analytical approach assuming constant specific fuel consumption and uncambered wing drag polar results in higher optimum Mach numbers and shorter ranges for constant-altitude–constant-airspeed flight of turbojet/fan transport aircraft. It is also shown that highly accurate approximate solutions can be derived to find optimum Mach numbers without neglecting wing camber and thrust specific fuel consumption dependence on Mach number.

Both approximate solutions with small angle and average Mach number imply that optimum Mach number is directly proportional with the minimum drag Mach number corresponding to initial cruise condition. However, this proportionality is a function of the cruise fuel weight fraction, dependence of the specific fuel consumption on the Mach number, and wing camber.

Only one assumption has been made for approximate solutions: constant drag polar parameters. However, this seems to be an adequate hypothesis when flying in long-range conditions, because the Mach number is then well below the drag divergence value.[5] Moreover, this is beyond the scope of this study.

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